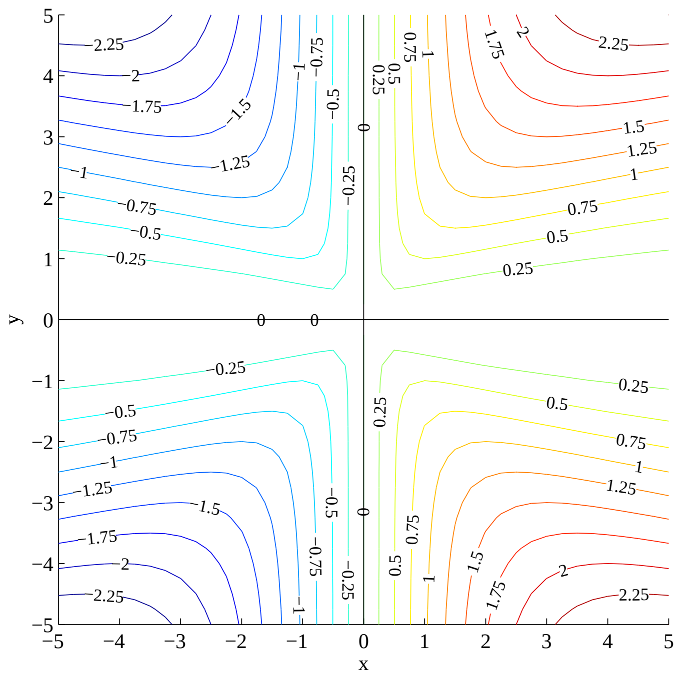
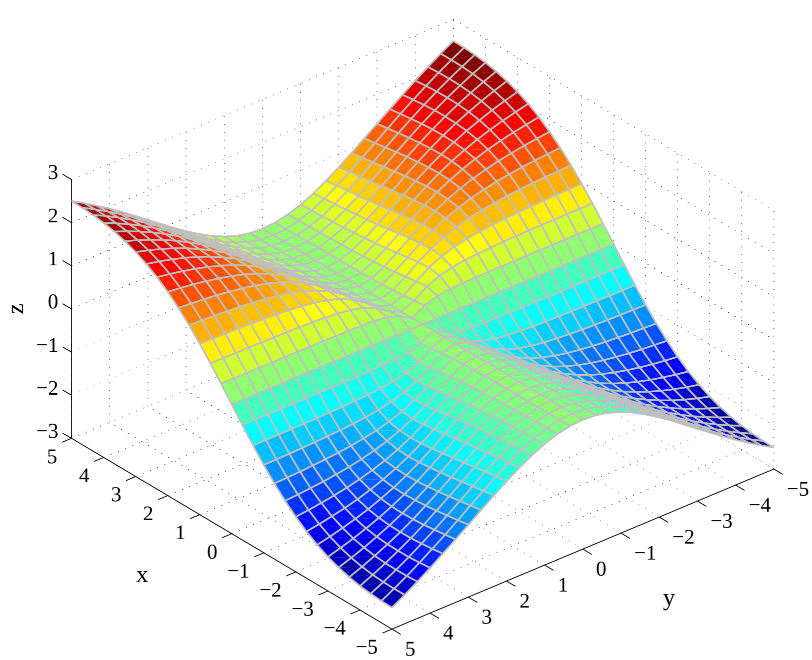
Differentiability

Consider the function given by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & ; \quad (x,y) \neq (0,0) \\[2ex] 0 & ; \quad (x,y) = (0,0) \end{cases}$$

The surface and the contour plot are shown below.





This function is continuous at $(0,0)$. This is true since

$$|f(x,y)| = \left|\frac{xy^2}{x^2+y^2}\right| \le \frac{|x|(x^2+y^2)}{x^2+y^2} \le |x|.$$

The Squeeze Theorem implies that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

The first partial derivatives are given by

$$f_x(x,y) = \frac{(x^2+y^2)y^2 - xy^2(2x)}{(x^2+y^2)^2} = \frac{x^2y^2+y^4-2x^2y^2}{(x^2+y^2)^2} = \frac{y^2(y^2-x^2)}{(x^2+y^2)^2},$$

$$f_y(x,y) = \frac{(x^2+y^2)(2xy) - xy^2(2y)}{(x^2+y^2)^2} = \frac{2x^3y+2xy^3-2xy^3}{(x^2+y^2)^2} = \frac{2x^3y}{(x^2+y^2)^2}.$$

Note that as long as $(x,y) \neq (0,0)$ these exist and are continuous. Therefore $f(x,y)$ is differentiable for all $(x,y) \neq (0,0)$. Also $f_x(0,y) = 1$ and $f_x(x,0) = 0$ and $f_y(x,0) = f_y(0,y) = 0$.

At the point $(0,0)$ we need to use the definition of the partial derivative to determine if f_x or f_y exist there. In this way we have

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

This shows that $f_x(0,0)$ and $f_y(0,0)$ exist, but are they continuous? Note

$$f_x(x,x) = 0, \quad f_x(x,\sqrt{x}) = \frac{x(x-x^2)}{(x^2+x)^2} = \frac{x^2(1-x)}{x^2(x+1)} = \frac{1-x}{1+x}.$$

Therefore $\lim_{x\to 0} f(x,x) = 0$, but $\lim_{x\to 0} f(x,\sqrt{x}) = 1$ implying that $f_x(x,y)$ is not continuous at $(0,0)$. Similarly

$$f_y(x,0) = 0, \quad f_y(x,x) = \frac{2x^4}{4x^4} = \frac{1}{2}.$$

Therefore $\lim_{x\to 0} f_y(x,0) = 0$, but $\lim_{x\to 0} f_y(x,x) = \frac{1}{2}$ implying that $f_y(x,y)$ is not continuous at $(0,0)$.

In order to show that f is not differentiable we need to use the definition of differentiable.

Definition A function f is differentiable at the point (a,b) if there is a linear function $L(x,y) = m(x-a) + n(y-b)$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - L(a+h,b+h)}{\sqrt{h^2+k^2}} = 0.$$

If f is differentiable then $m = f_x(a,b)$ and $n = f_y(a,b)$. For the function we are considering here we must have $m = f_x(0,0) = 0$ and $n = f_y(0,0) = 0$ and so $L(x,y) = 0(x-0)+0(y-0) = 0$. We must then look at the following expression.

$$\frac{f(h,k) - f(0,0) - L(h,k)}{\sqrt{h^2+k^2}} = \left(\frac{hk^2}{h^2+k^2}\right)\frac{1}{\sqrt{h^2+k^2}} = \frac{hk^2}{(h^2+k^2)^{3/2}}.$$

If $h = k$ this expression becomes

$$\frac{h^3}{(h^2+h^2)^{3/2}} = \frac{h^3}{(2h^2)^{3/2}} = \frac{h^3}{h^3 2^{3/2}} = \frac{1}{2^{3/2}}.$$

and if $h = 0$, $k \neq 0$ this expression is 0. This shows that

$$\lim_{(h,k)\to(0,0)} \frac{hk^2}{(h^2+k^2)^{3/2}} \quad \text{does not exist.}$$

Therefore f is not differentiable at $(0,0)$.
We do have that since $f_x(0,0) = f_y(0,0) = 0$ then $\nabla f(0,0) = 0\mathbf{i} + 0\mathbf{j}$. So if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is any unit vector the directional derivative at $(0,0)$ is

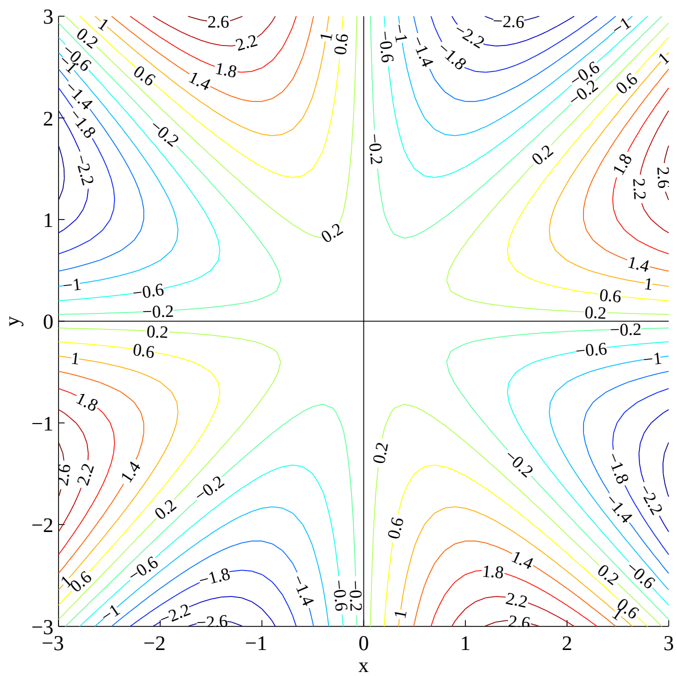
$$D_{\mathbf{u}}f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = 0(u_1) + 0(u_2) = 0.$$

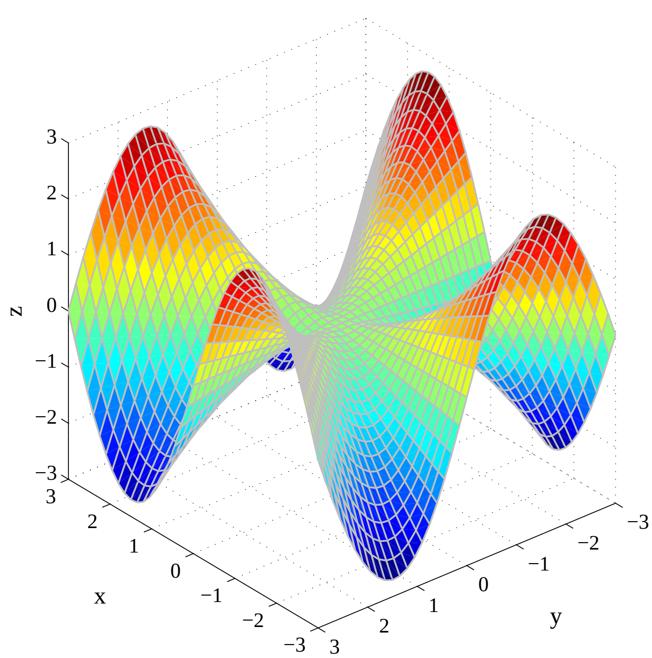
This shows that the directional derivative of f exists at $(0,0)$, but f is not differentiable at $(0,0)$.

Consider the function given by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & ; \quad (x,y) \neq (0,0) \\[2ex] 0 & ; \quad (x,y) = (0,0) \end{cases}$$

The surface and the contour plot are shown below.





This function is continuous at $(0,0)$ since

$$|f(x,y)| = \left|\frac{xy(x^2-y^2)}{x^2+y^2}\right| \le \frac{|xy|(x^2+y^2)}{x^2+y^2} = |xy|.$$

The Squeeze Theorem then implies that $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

The partial derivatives are then

$$\begin{aligned} f_x(x,y) &= \frac{(x^2+y^2)(3x^2-y^3)-(x^3y-xy^3)(3x)}{(x^2+y^2)^2} \\ &= \frac{3x^4y+3x^2y^3-x^2y^3-y^5-2x^4y+2x^2y^2}{(x^2+y^2)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} \\ &= \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}, \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= \frac{(x^2+y^2)(x^3-3xy^2)-(x^3y-xy^3)(2y)}{(x^2+y^2)^2} \\ &= \frac{x^5+x^3y^2-3x^3y^2-3xy^4-2x^3y^2+2xy^4}{(x^2+y^2)^2} \\ &= \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} \\ &= \frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} \end{aligned}$$

Both of these are continuous if $(x,y) \neq (0,0)$ and so f is differentiable for $(x,y) \neq (0,0)$. Note that $f_x(x,0) = 0$, $f_x(0,y) = \dfrac{-y^5}{y^4} = -y$, $f_y(x,0) = \dfrac{x^5}{x^4} = x$, $f_y(0,y) = 0$. At the point $(x,y) = (0,0)$ we use the definition of the partial derivatives.

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

and

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

Therefore $f_x(0,0) = f_y(0,0) = 0$. The partial derivatives are also continuous at $(0,0)$ since

$$
\begin{aligned}
|f_x(x,y)| &\le \frac{|y|(x^4+2x^2+y^2+y^4)}{(x^2+y^2)^2} + \frac{|y|(2x^2y^2)}{(x^2+y^2)^2} \\
&\le \frac{|y|(x^2+y^2)^2}{(x^2+y^2)^2} + \frac{2|y|(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^2} \\
&= 3|y|.
\end{aligned}
$$

The Squeeze Theorem then implies that $\lim_{(x,y)\to(0,0)} f_x(x,y) = 0 = f_x(0,0)$. A similar argument given below shows that $\lim_{(x,y)\to(0,0)} f_y(x,y) = 0 = f_y(0,0)$.

$$
\begin{aligned}
|f_y(x,y)| &\le \frac{|x|(x^4+2x^2y^2+y^4)}{(x^2+y^2)^2} + \frac{|x|(2x^2y^2)}{(x^2+y^2)^2} \\
&\le \frac{|x|(x^2+y^2)^2}{(x^2+y^2)^2} + \frac{2|x|(x^2+y^2)(x^2+y^2)}{(x^2+y^2)^2} \\
&= 3|x|.
\end{aligned}
$$

This function is quite special since

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k-0}{k} = -1,$$

and

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

This shows that the second mixed partial derivatives at $(0,0)$ in one order are different from the other order, i.e.

$$f_{xy}(0,0) = -1 \neq 1 = f_{yx}(0,0).$$