Central Michigan University
Department of Mathematics

STA 584 **Supplementary Examples** (not to be graded) Fall, 2003

1.

(a) If A and B are independent events, P(A) = .40 and P(B) = .70, find
(i) $P(A \cap B)$ (ii) $P(A \cup B)$ (iii) $P(A' \cap B)$

(b) Of the three events A, B and C, A and B are mutually exclusive, A and C are independent and B and C are independent. If P(A) = 1/4, P(B) = 1/3 and P(C) = 1/6, find
(i) $P(A \cup B)$ (ii) $P(A \cup C)$ (iii) $P(A \cup B \cup C)$

2. A certain examination is such that 80% of the in-state students pass and only 25% of the out-of-state students pass. Furthermore, 70% of the students are in-state.

(a) What is the probability that a student selected at random will pass the examination?
(b) What is the probability that a student who passes is in fact an in-state student?
(c) What is the probability that a successful student selected at random will be an out-of-state student?

3.

(a) Among the five nominees for two vacancies on a city council are two men and three women.
(i) In how many ways can these vacancies be filled with any two of the five nominees;
(ii) What is the probability that the vacancies will be filled with one of the men and one of the women?

(b) Suppose A and B are events such that P(A) = 0.7, P(B) = 0.2, and $P(A \cap B) = 0.1$.
(i) Compute: $P(A' \cap B')$; $P(A \cap B')$; $P(A \cup B)$; $P(A' \cup B)$; $P(A'|B)$.
(ii) Are A and B mutually exhaustive? Justify your answer.

(c) Two methods, A and B, are available for teaching a certain industrial skill. The failure rate is 20% for A and 10% for B. However, B is more expensive and hence is used only 30% of the time (A is used the other 70%).
(i) What is the probability that a worker failed to learn correctly?
(ii) If a randomly selected worker failed to learn the skill correctly, what is the probability that the worker was taught by method A ?

4. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - kx, & 1 \le x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Evaluate the constant k.
(b) Find the cumulative distribution function for X.
(c) Find the following probabilities: (i) $P(X < 0.5)$ (ii) $P(X > 1.5)$ (iii) $P(0.8 < X < 1.2)$

5. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} kx^3, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Evaluate the constant k. (b) Find the distribution function of X.

(c) Find the following probabilities: (i) $P(X < \frac{1}{2})$ (ii) $P(1/4 < X < \frac{1}{2})$

(iii) $P(X < \frac{3}{4})$ (iv) $P(X < 2)$

6. The joint probability distribution of X and Y is given by the following table.

		x = -1	x = 0	x = 1	x = 3
	-1	.02	.01	.02	.11
y	2	.06	.05	.06	.15
	3	.11	.10	.11	.20

(a) Find: (i) $P(X \le 1, Y > 2)$ (ii) $P(X = 0, Y \le 2)$ (iii) $P(X + Y > 2)$

(iv) $P(X^2 + Y^2 \le 5)$

(b) Find the marginal distribution of Y.

(c) Find the marginal distribution of X.

(d) Are X and Y independent? Justify your answer.

7. Suppose X and Y are continuous random variables with joint probability density function

$$f(x, y) = \begin{cases} kxy, & 0 < x < 1,\ 0 < y < 1,\ x + y > 1 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find (a) the value of constant k;

(b) $P(X + Y > \frac{3}{2})$; (c) $P(X \ge \frac{1}{2}, Y \ge \frac{3}{4})$;

(d) the marginal distribution of X; (e) the marginal distribution of Y;

(f) the conditional distribution of Y given that X = x.

(ii) Are X and Y independent? Justify your answer.

8. Let X and Y denote the proportions of time, out of one workday, that employees A and B, respectively, actually spend performing their assigned tasks. The joint relative frequency behavior of X and Y is modeled by the density function

$$f(x, y) = \begin{cases} \dfrac{x + y}{k}, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of k that makes this function a probability density function.

(b) Find $P(X \le \frac{1}{4}, Y > \frac{1}{4})$ (c) Find $P(X + Y \le 1)$

(d) Find the marginal distribution of Y.

(e) Find the marginal distribution of X.

(f) Are X and Y independent? Explain. (g) Find $P(X \le \frac{1}{2} \mid Y \le \frac{1}{2})$.

(h) If employee B spends exactly 50% of the day on assigned duties, find the probability that employee A spends less than 75% of the day on similar duties?

9.

(a) Show that if a random variable X has a probability density function

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

the probability that it will take on a value more than $b - p(b - a)$ is equal to p.

(b) Find the moment generating function of the random variable X in (a).

(c) Using your result in (b) or otherwise, find E(X).

10. A random variable X has a beta density function

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

(a) Show that if $\alpha > 1$ and $\beta > 1$, the beta density has a relative maximum at $x = (\alpha - 1)/(\alpha + \beta - 2)$. What happens when (i) $\alpha = 1$; $\beta > 1$ and (ii) $\alpha = 1$; $\beta = 1$?

(b) Find the mean of X.

11. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} 5e^{-5x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the moment generating function of the random variable X.

(b) Using the result in (a) or otherwise, find (i) E(X) (ii) Var(X) (iii) $E(X^3)$

(c) Find $E(e^{3X})$

(d) If Y = 10X + 3, find the moment generating function of Y.

(e) Using the result in (d) or otherwise, find E(Y) and Var(Y).

12. A gas station operates two pumps, each of which can pump up to 10,000 gallons of gas in a month. The total amount of gas pumped at the station in a month is a random variable X (measured in ten thousands of gallons), which has a probability density function given by

$$f(x) = \begin{cases} kx, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of k.

(b) Find the mean of the distribution.

(c) Find the cumulative distribution function F(x).

(d) Find the probability that the station pumps between 8,000 and 12,000 gallons in a month.

(e) Given that the station has pumped over 10,000 gallons for a particular month, find the probability that the station pumped over 15,000 gallons during the month.

13.

(a) Let X have the density function

$$f(x) = \begin{cases} 4xe^{-2x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the moment generating function for X.
(ii) Find the mean and variance of X.

(b) Find the distribution of the random variable Y for each of the following moment generating functions:

(i) $M_Y(t) = (\frac{2}{3} + \frac{1}{3}e^t)^5$
(ii) $M_Y(t) = \exp(2(e^t - 1))$
(iii) $M_Y(t) = (1 - \theta t)^{-1}$

(c) Let $M_X(t) = (1 - \theta)e^t (1 - \theta e^t)^{-1}$. Find the mean and the variance of X.

14.

(a) Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{1}{2}e^{-x/2}, & x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the mean and the variance of the random variable X.
(ii) Calculate $P(X > 3)$
(iii) Calculate $P(X > 5 \mid X > 2)$

(b) The proportion of time per day that all checkout counters in a certain supermarket are busy is a random variable Y, with density function given by

$$f(y) = \begin{cases} cy^2(1-y)^4, & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the value of c that makes f(y) a probability density function.
(ii) Find E(Y) and Var(Y).

15. Weekly CPU time used by an accounting firm has a probability density function (measured in hours) given by

$$f(x) = \begin{cases} \dfrac{3x^2(4-x)}{64}, & 0 \leq x \leq 4 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the expected value and variance of weekly CPU time.

(b) The CPU time costs the firm \$200.00 per hour. Find the expected value and variance of the weekly cost for CPU time.

(c) Would you expect the weekly cost to exceed \$600.00 very often? Why?

16. The length of time that is required by students to complete a 1-hour examination is a random variable with a density function given by

$$f(x) = \begin{cases} cx^2 + x, & 0 \le x \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find c. (b) Find the cumulative distribution function F(x).
(c) Find the probability that a student finishes in less than a half-hour.
(d) Given that a student needs at least 15 minutes to complete the examination, find the probability that she will require at least 30 minutes to finish.
(e) Find the mean of X. (f) Find the variance of X and hence the standard deviation of X.

17. An agricultural cooperative claim that 95% of the watermelons shipped out are ripe and ready to eat. Let X represent the number of watermelons that are ripe and ready to eat among eight that are shipped out.
(a) Find the probabilities that (i) all eight are ripe and ready to eat;
(ii) at least six are ripe and ready to eat; (iii) at most 2 are ripe and ready to eat;
(iv) between two and five, not inclusive, are ripe and ready to eat.
(b) On the average, how many of the watermelons are ripe and ready to eat?
(c) Find the standard deviation of X.
(d) What proportion of the ripe and ready to eat watermelons lies between two standard deviations of the mean?

18. A warehouse contains 10 printing machines, 4 of which are defective. A company selects 5 of the machines at random, thinking all are in working condition
(a) What is the probability that
(i) none is defective? (ii) at most 1 is defective?
(iii) at least 4 are defective? (iv) exactly 3 are defective?
(b) By using an appropriate discrete distribution, approximate the probability in (iv).
(c) What is the error of approximation in (b)?
(d) Is the approximation in (b) justified? Why or why not?

19. Let X be a discrete random variable with the Poisson distribution

$$P(X = x) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0,\ 1,\ 2,\ \ldots$$

(a) Find the moment generating function of the random variable X.
(b) Differentiating with respect to θ the expressions on both sides of the equation $\sum_{x=0}^{\infty} \frac{\theta^x e^{-\theta}}{x!} = 1$, show that the mean of the Poisson distribution is given by θ. Differentiating again with respect to θ, show that $\mu_2' = \theta + \theta^2$ and hence $\sigma^2 = \theta$.
(c) In a particular department store customers arrive at a checkout counter according to a Poisson distribution at an average of 4 per half hour. During a given half hour, what is the probability that 2 customers arrive?

20. The probability that a patient recovers from a stomach disease is 0.8. Suppose twenty people are known to have contracted this disease and let X, a binomial random variable, denote the number that recovers.

(a) What is the probability that exactly 18 survive?

(b) What is the probability that at least 19 survive?

(c) Using normal approximation, compute the probability in (b).

(d) What is the error of approximation? Is the approximation justified? Explain.

21. The number of people, X, entering the intensive care unit at a particular hospital on any single day possesses a Poisson distribution with mean equal to four persons per day.

(a) What is the probability that the number of people entering the intensive care unit on a particular day is equal to 2?

(b) Is it likely that X will exceed 10? Explain.

(c) What is the probability that the number of people entering the intensive care unit on a particular half-day is equal to 2?

22. The width of bolts of fabric is normally distributed with a mean of 950 millimeters and a standard deviation of 10 millimeters.

(a) What is the probability that a randomly chosen bolt has a width

(i) more than 960 millimeters?

(ii) between 947 and 958 millimeters?

(iii) less than 945 millimeters?

(iv) equal to 970 millimeters?

(b) Find C such that a randomly chosen bolt has a width less than C with probability 0.8531.

23. Suppose X has a normal distribution and it is known that $P(X < 2.0) = 0.3085$ and $P(X > 3.0) = 0.2266$. Find the mean μ and the standard deviation σ for X.

24.

(a) Suppose that the annual rainfall in a certain region is a uniform random variable with parameters a = 12 inches and b = 15 inches. Find the probability that in a given year the region's rainfall will be

(i) less than 13 inches. (ii) less than 13 inches 3 out of 4 years.

(b) Suppose that the operating lifetime of a certain type of battery is an exponential random variable with parameter $\theta = 2$ (measured in years). Find the probability that

(i) a battery of this type will have an operating lifetime of over 4 years;

(ii) at least one out of 5 batteries of this type will have operating lifetime of over 4 years.

25. A manufacturer of electric light bulbs finds that, on the average, 5% of the bulbs are defective. Suppose a random sample of 475 bulbs is selected.

(a) What is the exact probability that more that 14 bulbs will be defective? Do not evaluate your result.

(b) Using an appropriate discrete distribution approximation, write down the probability in (a). Do not evaluate your result.

(c) Using an appropriate continuous distribution approximation, compute the probability in (a).

26. An aptitude test administered to aircraft pilot trainees requires a series of operations to be performed in succession. Suppose that the time needed to complete the test is normally distributed with mean 90 minutes and standard deviation 20 minutes.

(a) To pass the test, a candidate must complete it within 80 minutes.

(i) What percentage of the candidates will pass the test?

(ii) If 5 candidates take the test, find the probability that exactly one of them will pass.

(iii) What assumptions did you make in (ii)?

(b) If the top 10% of the candidates are to be given a certificate of commendation, how fast must a candidate complete the test to be eligible for a certificate?

27. The joint distribution for the length of life of two different types of components operating in a system is given by

$$f(x,y) = \begin{cases} \frac{x}{8} e^{-(x+y)/2}, & x > 0,\ y > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

The relative efficiency of the two types of components is measured by $W = \frac{Y}{X}$. Find the probability density function for W.

28.

(a) Let X be a random variable with the probability density function

$$f(x) = \begin{cases} 2(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the density function of $Y = 2X - 1$. (ii) Find the density function of $Z = 1 - 2X$.

(iii) Find E(Y) and E(Z) by using the derived density functions (for the random variables Y and Z) in (i) and (ii).

(iv) Find E(Y) and E(Z) by using the density function of X. Compare your results with those in (iii).

(b) Let X_1 be binomial random variable with parameters n_1 and θ. Let X_2 be another binomial random variable with parameters n_2 and θ. If X_1 and X_2 are independent, find the probability function of $Y = X_1 + X_2$. (Hint: Use moment generating function technique).

29. If the joint density function of the random variables X and Y is given by

$$f(x, y) = \begin{cases} 3x, & 0 < x < 1,\ 0 < y < 1,\ x + y \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal density function of X.
(b) Find the conditional density function of Y given that X = x.
(c) Find the joint density function of Z = X + Y and W = X.
(d) From your result in (c), find the density function of Z.

30. The amount of flour used per day by a bakery is a random variable X possessing a probability density function

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

The cost of the flour is proportional to Y = 3X + 1.
(a) Find the moment generating function for X.
(b) Find the probability density function for Y.
(c) Using the answer in (b) or otherwise, find E(Y).
(d) Find the probability that X is at most 2.
(e) Using your result in (a) find the moment generating function for Y.

Chapter	**Problems** (note that some problems cover more than one chapter)
2	1, 2, 3
3	4, 5, 6, 7, 8.
4	9, 10, 11, 12, 13, 14, 15, 16.
5	17, 18, 19, 20, 21.
6	22, 23, 24, 25, 26.
7	27, 28, 29, 30.