1. (#8, p. 219 (217)) For part (a), we use the known expansion for $e^x$ and replace $x$ by $x^2$ to obtain
$$e^{x^2} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k}.$$
Now we integrate on both sides from 0 to 1, using the fact that we can integrate a power series term-by-term and evaluating the resultant integrals to get
$$\int_0^1 e^{x^2} dx = \sum_{k=0}^{\infty} \frac{1}{(2k+1)k!}.$$
Hence, for any positive integer $n$ we have
$$\begin{aligned} \int_0^1 e^{x^2} dx - \sum_{k=0}^{n-1} \frac{1}{(2k+1)k!} &= \sum_{k=n}^{\infty} \frac{1}{(2k+1)k!} \\ &\leq \frac{1}{n!} \sum_{k=0}^{\infty} \frac{1}{k!} \\ &< \frac{3}{n!}. \end{aligned}$$
The desired numerical inequalities now follow by taking $n = 9$.

2. (#9, p. 219 (217)) First suppose that $f$ is analytic on $(a, b)$. Let $x_0 \in (a, b)$. By assumption, there is an interval $(x_0 - R, x_0 + R)$ contained in $(a, b)$ so that
$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$
for $|x - x_0| < R$. Since power series may be differentiated term-by-term, we have
$$f'(x) = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}$$
for $|x - x_0| < R$ which says that $f'$ is analytic at $x_0$, and hence on $(a, b)$ since $x_0$ was arbitrary. On the other hand, if $f'$ is analytic on $(a, b)$, then there is some $R$ with $(x_0 - R, x_0 + R)$ contained in $(a, b)$ such that
$$f'(x) = \sum_{k=0}^{\infty} b_k (x - x_0)^k$$

for $|x - x_0| < R$. For any $x$ satisfying this inequality, we have (since power series may be integrated term-by-term),

$$f(x) = \int_{x_0}^{x} f'(t)dt = \sum_{k=0}^{\infty} \frac{b_k}{k+1}(x - x_0)^{k+1},$$

which shows $f$ to be analytic.

3. (#3, hand-out sheet). (a) For $x \in [c, d]$ with $c > 0$ we have

$$|e^{-ax} \cos x| \leq e^{-cx}$$

and since the integral from 0 to $\infty$ of the function on the right converges (to $1/c$), the original integral converges, as indicated, to $a/a^2 + 1)$ uniformly on $[c, d]$ as a result of the $M$-test for integrals. (This is from Problem 2(b) with $b = 1$.) Hence we can integrate both sides from $c$ to $d$ (with respect to $a$). On the left side, using the fact that we can interchange order of the integrals because of the uniform convergence, we obtain

$$\int_c^d \int_0^\infty e^{-ax} \cos x dx da = \int_0^\infty \int_c^d e^{-ax} \cos x da dx$$
$$= \int_0^\infty \frac{e^{-cx} - e^{-dx}}{x} \cos x dx.$$

The result of integration on the other side is

$$\int_c^d \frac{a}{a^2+1} da = \frac{1}{2} \log\left(\frac{d^2+1}{c^2+1}\right).$$

If we now replace $c$ by $a$ and $d$ by $b$, we get the statement to be proved. Here we want to differentiate the equation of part 2(a)(on the handout sheet) on both sides with respect to $b$ and move the differentiation with respect to $b$ inside the integral. This is allowed if we first show that the integral

$$\int_0^\infty -xe^{-ax} \sin(bx) dx$$

converges uniformly with respect to $b$. This follows since the absolute value of the integrand is uniformly bounded by $xe^{-ax}$ on the whole real line, and the integral

$$\int_0^\infty xe^{-ax}$$

converges as long as $z > 0$. The desired equality now follows since the derivative of $a/a^2 + b^2)$ with respect to $b$ is $-2ab/(a^2 + b^2)^2$.

4. (#6, hand-out sheet). Since $f$ is piecewise continuous and bounded on each finite interval, we may actually assume that

$$|f(t)| \leq Me^{at}$$

for all $t \geq 0$. Now we have

$$|f(t)e^{-st}| \leq Me^{(a-s)t} \leq Me^{(a-a_1)t}$$

for all $s \geq a_1 > a$. Since the integral of the function on the right is finite, the given integral converges uniformly by the $M$-test. Similarly, the integral $\int_o^\infty te^{-st}f(t)dt$ convergences uniformly by comparison with the integral $\int_0^\infty tMe^{(a-a_1)t}dt$. The statement about $\varphi'(s)$ follows from this. Finally,

$$|\varphi(s)| \leq \int_0^\infty Me^{(a-s)t}dt = \frac{M}{s-a}$$

which shows that $\varphi(s) \to 0$ as $s \to \infty$.

5. (#5, p. 512 (499)) (a) Observe that for any $N$,

$$|a_k(f_N) - a_k(f)| \leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f_N(x) - f(x)|dx,$$

so given $\epsilon > 0$, choose $N_0$ so large that

$$|f_N(x) - f(x)| < \frac{\epsilon}{2}$$

for all $x \in [-\pi, \pi]$ whenever $N \geq N_0$. It follows that the integral above is less than $\epsilon$ for large $N$. The argument for $b_k(f_N)$ is clearly the same.

(b) It is clear from the above argument that is really needed is that the integral

$$\int_{-\pi}^{\pi} |f_N(t) - f(t)|dt < \frac{\epsilon}{\pi}.$$

This is true for large $N$ by the hypothesis.

6. (#6a,b), p. 512 (499)). (a) Straightforward integration shows that $a_k(f) = 0$ for all $k$ and $b_k(f) = 0$ for $k$ even, and $4/(k\pi)$ when $k$ is odd.

(b)Note that $S_{2N}(f) = S_{2N-1}(f)$. Now

$$S_{2N}(f) = \sum_{k=1}^{N} \frac{4}{\pi(2k-1)} \sin(2k-1)x$$

so that

$$S'_{2N}(f) = \sum_{k=1}^{N} \frac{4}{\pi} \cos(2k-1)x.$$

Using the identity $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$, we obtain

$$\begin{aligned} \sin x S'_{2N}(f) &= \sum_{k=1}^{N} \frac{4}{\pi} \sin x \cos(2k-1)x \\ &= \sum_{k=1}^{N} \frac{2}{\pi} [\sin(2k)x - \sin(2k-2)x] \\ &= \sin(2N)x \end{aligned}$$

Dividing both sides by $\sin x$ and integrating from 0 to $x$ yields the desired equality.